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# THE TORIC IDEAL OF A GRAPHIC MATROID IS GENERATED BY QUADRICS

JONAH BLASIAK

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Describing minimal generating sets of toric ideals is a well-studied and difficult problem. Neil White conjectured in 1980 that the toric ideal associated to a matroid is generated by quadrics corresponding to single element symmetric exchanges. We give a combinatorial proof of White's conjecture for graphic matroids.

## 1. Introduction

Let $M$ be a matroid on the ground set $\{1,2,\dots,m\}$. Fix a field $k$ and define the polynomial ring $S_M$ to be $k[y_B : B \text{ a base of } M]$. The toric ideal $I_M$ associated to $M$ is the kernel of the $k$-algebra homomorphism $\theta_M : S_M \to k[x_1,\dots,x_m]$ that takes $y_B$ to $\prod_{i\in B} x_i$.

Given bases $B$ and $D$ of $M$, the well-known *symmetric exchange property* states that for every $b\in B$ there exists a $d\in D$ such that $B\cup d-b$ and $D\cup b-d$ are bases. We say that $b\in B$ *double swaps* into $D$; if $B\cup d-b$ and $D\cup b-d$ are bases, we say that $b\in B$ and $d\in D$ double swap. Neil White made a conjecture in [7] about an equivalence relation defined by certain symmetric exchange properties and we state an algebraic reformulation similar to the one given in [5].

**Conjecture 1.1.** For any matroid $M$, the toric ideal $I_M$ is generated by the quadratic binomials $y_{B_1}y_{B_2}-y_{D_1}y_{D_2}$ such that the pair of bases $D_1, D_2$ can be obtained from the pair $B_1, B_2$ by a double swap.

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The cycle matroid of a graph $G$, which we denote by $M(G)$, is the matroid on the ground set $E(G)$ with a base for each spanning forest of $G$. A matroid is said to be *graphic* if it is the cycle matroid of some graph. We prove White's conjecture for graphic matroids.

**Theorem 1.2.** *If $M$ is a graphic matroid, then the toric ideal $I_M$ is generated by the quadratic binomials $y_{B_1}y_{B_2} - y_{D_1}y_{D_2}$ such that the pair of bases $D_1, D_2$ can be obtained from the pair $B_1, B_2$ by a double swap.*

The arguments given in this paper are almost all combinatorial, but White's conjecture is more natural in its algebraic formulation than its combinatorial one. We will briefly mention how White's conjecture fits in with other results and conjectures about toric ideals and varieties. Let $b$ be the number of bases of $M$. The set of vectors in $k^b$ that vanish on all polynomials in the toric ideal $I_M$ is the affine toric variety $\mathrm{Spec}(S_M/I_M)$. For each matroid $M$, the ideal $I_M$ is homogeneous because every base has the same number of elements. Therefore $I_M$ also defines a projective toric variety $Y_M = \mathrm{Proj}(S_M/I_M)$ in $\mathbb{P}_k^{b-1}$.

Given $\mathbf{u} \in \mathbb{Z}^b$ define $\mathbf{u}_+$ (resp. $\mathbf{u}_-$) to be $\mathbf{u}$ (resp. $-\mathbf{u}$) with negative coordinates replaced by zeros; we then have $\mathbf{u} = \mathbf{u}_+ - \mathbf{u}_-$. The difference of two monomials is a *binomial*. From the point of view of Sturmfels in [5], toric ideals are ideals generated by binomials $\mathbf{y}^{\mathbf{u}_+} - \mathbf{y}^{\mathbf{u}_-}$, where $\mathbf{u}$ runs over integer vectors in the kernel of an integer matrix. For the toric ideal $I_M$, the integer matrix is the $m \times b$ matrix whose columns are the zero-one incidence vectors of the bases of $M$. In algebraic geometry, it is sometimes required that toric ideals have an associated affine toric variety (resp. projective toric variety) that is normal (resp. projectively normal). The ideals $I_M$ are toric in this sense too by the following theorem of White [6].

**Theorem 1.3.** *For any matroid $M$, the toric variety $Y_M$ is projectively normal.*

The following is a general theorem about projectively normal toric varieties due to Hochster (see e.g. [5]).

**Theorem 1.4.** *If the toric ideal $I$ defines a projectively normal $r$-dimensional toric variety, then $I$ is generated by binomials of degree at most $r$.*

This conjecture restricted to toric varieties coming from matroids is much weaker than White's conjecture, but the degree bound is sharp for general toric ideals [5].

The analog of Theorem 1.4 for Gröbner bases is known in several special cases, but remains open in general [5].

**Conjecture 1.5.** If the toric ideal $I$ defines a projectively normal $r$-dimensional toric variety, then $I$ has a Gröbner basis consisting of binomials of degree at most $r$.

It is also natural to ask whether the following variant of White's conjecture holds (see, for instance, [3] and chapter 14 of [4]).

**Conjecture 1.6.** For any matroid $M$, the toric ideal $I_M$ has a Gröbner basis consisting of quadratic binomials.

White's conjecture can be posed as two separate conjectures. The following are both still open and together imply White's conjecture.

**Conjecture 1.7.** For any matroid $M$, the toric ideal $I_M$ is generated by quadratic binomials.

**Conjecture 1.8.** For any matroid $M$, the quadratic binomials of $I_M$ are in the ideal generated by the binomials $y_{B_1}y_{B_2} - y_{D_1}y_{D_2}$ such that the pair of bases $D_1, D_2$ can be obtained from the pair $B_1, B_2$ by a double swap.

Sturmfels shows in chapter 14 of [4] that Conjecture 1.6 holds for uniform matroids. One may also ask the same questions about toric ideals coming from polymatroids. Conca proves Conjecture 1.7 for transversal polymatroids [2]. Caviglia, Elizalde, and García prove that both White's conjecture and Conjecture 1.6 hold for a certain class of polymatroids they call staircase polymatroids [1].

In Section 2 we show that Conjecture 1.7 holds for graphic matroids if certain graphs $\mathfrak{G}_n(M)$ are connected for all $n \geq 3$ and all graphic matroids $M$. Similarly, Conjecture 1.8 holds for graphic matroids if the graph $\mathfrak{G}(M)$ is connected for every graphic matroid $M$. In Section 3 we prove that the graphs $\mathfrak{G}_n(M)$ are connected for any graphic matroid $M$. In Section 4 we prove that the graph $\mathfrak{G}(M)$ is connected for any graphic matroid $M$. In section 5 we discuss the difficulties of extending our results to general matroids and pose some questions along these lines.

## 2. Reduction

We show that the algebraic formulation of White's conjecture is implied by a combinatorial condition similar to White's original formulation.

Let $M$ be a matroid on a ground set of size $r(M)n$, where $r(M)$ denotes the rank of $M$. The *$n$-base graph* of $M$, which we denote by $\mathfrak{G}_n(M)$, has as its vertex set the set of all sets of $n$ disjoint bases (a set of $n$ bases $\{B_1, \dots, B_n\}$

of $M$ is disjoint if and only if $\bigcup_i B_i$ is the entire ground set of $M$). There is an edge between $\{B_1,\dots,B_n\}$ and $\{D_1,\dots,D_n\}$ if and only if $B_i = D_j$ for some $i,j$. We prove that Conjecture 1.7 is implied by the connectivity of the $n$-base graphs. We prove the following proposition for a general class of matroids $\mathfrak{C}$ that is closed under deletions and adding parallel elements, but we will only apply this to the case where $\mathfrak{C}$ is the set of graphic matroids.

**Proposition 2.1.** *Let $\mathfrak{C}$ be a collection of matroids that is closed under deletions and adding parallel elements. Suppose that for each $n \geq 3$ and for every matroid $M$ in $\mathfrak{C}$ on a ground set of size $r(M)n$ the $n$-base graph of $M$ is connected. Then for every matroid $M$ in $\mathfrak{C}$, $I_M$ is generated by quadratic binomials.*

**Proof.** We will prove by induction on $n$ the statement that for every $M \in \mathfrak{C}$ and every binomial $b \in I_M$ of degree $n$, $b$ is in the ideal generated by the quadrics of $I_M$. This will prove the proposition because, as mentioned in the introduction, $I_M$ is spanned as a $k$-vector space by binomials. For the base case $n = 2$ there is nothing to prove. Suppose $n \geq 3$, $M$ is a matroid in $\mathfrak{C}$ on the ground set $\{1, 2, \dots, m\}$, and $b$ is a binomial of degree $n$ in $I_M$. The binomial $b$ is necessarily of the form $b = \prod_{i=1}^n y_{B_i} - \prod_{i=1}^n y_{D_i}$ for some bases $B_1,\dots,B_n, D_1,\dots,D_n$ of $M$ such that the $B_i$ and $D_i$ have the same multiset union. We will show that $b$ is in the ideal generated by the degree $n-1$ binomials of $I_M$ by constructing a new matroid $M'$ that depends on the binomial $b$. By induction, the degree $n-1$ binomials are in the ideal generated by the quadrics of $I_M$ so this will complete the proof.

Put $\mathbf{x}^{\mathbf{S}} = \theta_M(\prod_{i=1}^n y_{B_i})$ and let $\mathbf{S}_i$ denote the $i^{th}$ component of $\mathbf{S}$. Define $M'$ to be the matroid obtained from $M$ by replacing $i$ with $\mathbf{S}_i$ parallel copies of $i$ for each $i$ in $\{1,\dots,m\}$; interpret "replacing by zero parallel copies" to mean deleting, that is, delete those $i$ for which $\mathbf{S}_i = 0$. There is a natural map $\alpha$ from the ground set of $M'$ to the ground set of $M$ that takes each of the parallel copies of $i$ to $i$. If $X$ is an independent set of $M'$, then $\alpha(X)$ is an independent set of $M$. So there is a $k$-algebra homomorphism $\alpha_* : S_{M'} \to S_M$ defined by $\alpha_*(y_{B'}) = y_{\alpha(B')}$ for every base $B'$ of $M'$.

Because the collection $\mathfrak{C}$ is closed under deletions and adding parallel elements, $M \in \mathfrak{C}$ implies $M' \in \mathfrak{C}$. $M'$ has a ground set of size $r(M')n = r(M)n = \sum_i \mathbf{S}_i$, and by assumption, the $n$-base graph of $M'$ is connected. Let $\mathbf{u_B}$ be a vertex of $\mathfrak{G}_n(M')$ such that $\alpha(\mathbf{u_B}) = \{B_1,\dots,B_n\}$ (here $\alpha$ is the natural extension of $\alpha$ to sets of subsets of the ground set of $M'$: $\alpha(\mathbf{u_B}) = \{\alpha(X) | X \in \mathbf{u_B}\}$). Such a $\mathbf{u_B}$ exists by construction of $M'$: simply split up the parallel copies of $i$, giving one to each base in $\{B_1,\dots,B_n\}$ containing $i$. Let $\mathbf{u_D}$ be a vertex of $\mathfrak{G}_n(M)$ such that $\alpha(\mathbf{u_D}) = \{D_1,\dots,D_n\}$.

Let $\mathbf{y}^{\mathbf{u}} = \prod_{X \in \mathbf{u}} y_X$, as is customary when $\mathbf{u}$ is identified with its zero-one incidence vector. Let $\mathbf{u_0}, \mathbf{u_1}, \dots, \mathbf{u_t}$ be the vertices of a path between $\mathbf{u_B} = \mathbf{u_0}$ and $\mathbf{u_D} = \mathbf{u_t}$ in $\mathfrak{G}_n(M')$. Then we have

$$\sum_{i=1}^{t} \mathbf{y}^{\mathbf{u_{i-1}}} - \mathbf{y}^{\mathbf{u_i}} = \mathbf{y}^{\mathbf{u_0}} - \mathbf{y}^{\mathbf{u_t}}$$

and applying the map $\alpha_*$ we obtain

$$(1) \qquad \sum_{i=1}^{t} \mathbf{y}^{\alpha(\mathbf{u_{i-1}})} - \mathbf{y}^{\alpha(\mathbf{u_i})} = \mathbf{y}^{\alpha(\mathbf{u_0})} - \mathbf{y}^{\alpha(\mathbf{u_t})} = \prod_{i=1}^{n} y_{B_i} - \prod_{i=1}^{n} y_{D_i} = b.$$

For $i = 1, \dots, t$ there is a base $X \in \mathbf{u_{i-1}} \cap \mathbf{u_i}$ which implies $\alpha(X) \in \alpha(\mathbf{u_{i-1}}) \cap \alpha(\mathbf{u_i})$. This shows that $y_{\alpha(X)}$ may be factored out of the binomial $\mathbf{y}^{\alpha(\mathbf{u_{i-1}})} - \mathbf{y}^{\alpha(\mathbf{u_i})}$, and therefore (1) shows that $b$ is in the ideal generated by the degree $n-1$ binomials of $I_M$. ∎

The reduction for k̆Conjecture 1.8 is similar. Suppose $M$ is a matroid on a ground set of size $2r(M)$. The *single exchange graph* of $M$, which we denote by $\mathfrak{G}(M)$, is the graph with vertex set the set of ordered 2-tuples of bases of $M$, $(B_1, B_2)$, such that $B_1$ and $B_2$ are disjoint. There is an edge between $(B_1, B_2)$ and $(D_1, D_2)$ if and only if $(D_1, D_2)$ can be obtained from $(B_1, B_2)$ by a double swap (the order of the bases matters, that is, we require $|B_1 \cap D_1| = |B_2 \cap D_2| = r(M) - 1$). The above proposition can be easily modified to show that: if for every $M$ in $\mathfrak{C}$ with a ground set of size $2r(M)$ the single exchange graph of $M$ is connected, then Conjecture 1.8 holds for all matroids in $\mathfrak{C}$.

**Remark 2.2.** Showing that the single exchange graph is connected actually shows slightly more than Conjecture 1.8. In $\mathfrak{G}(M)$, $(B_1, B_2)$ is not adjacent to $(B_2, B_1)$ for ranks larger than 1, however $y_{B_1} y_{B_2} - y_{B_2} y_{B_1} = 0$ is (trivially) in the ideal generated by quadrics corresponding to single double swaps. The stronger statement we prove here was also conjectured by White in [7].

## 3. Proof of the graphic case

We introduce some notation that is used in the main proof. Let $G$ be a graph. The sets $V(G)$ and $E(G)$ are the vertices and edges of $G$. If $v, v' \in G$, we abuse notation slightly and say that $v$ *is connected to* $v'$ or $v$ *and* $v'$ *are connected* to mean that $v$ and $v'$ are in the same component. We will use $d(v)$ for the degree of $v$. We use $-$ to denote set minus and sometimes write a one element set as the element itself rather than the element with braces around it. The notation $[n]$ is shorthand for $\{1,\dots,n\}$.

The following theorem together with Proposition 2.1 implies that Conjecture 1.7 holds for graphic matroids.

**Theorem 3.1.** *If $M$ is a graphic matroid of rank $r$ on a ground set of size $nr$ and $n \geq 3$, then the $n$-base graph $\mathfrak{G}_n(M)$ is connected.*

**Proof.** We prove the theorem by induction on $r$. Let $G$ be a graph such that $M = M(G)$. If $r = 1$, then $\mathfrak{G}_n(M)$ is empty or a single vertex, which of course is connected. Now suppose $r > 1$. First observe that we can assume $G$ is connected. An argument for this is given in the proof of Proposition 5 of [7], but we repeat it here for completeness. If $G$ is not connected, we may write $M(G)$ as the direct sum of $M(G_1)$ and $M(G_2)$, where $G_1$ and $G_2$ are unions of connected components of $G$. By the inductive hypothesis, the $n$-base graphs of $M(G_1)$ and $M(G_2)$ are connected. We can write any two vertices of $\mathfrak{G}_n(M)$ as $\{B_1^1 \sqcup B_1^2, \dots, B_n^1 \sqcup B_n^2\}$ and $\{D_1^1 \sqcup D_1^2, \dots, D_n^1 \sqcup D_n^2\}$ so that $\{B_1^i, \dots, B_n^i\}$ and $\{D_1^i, \dots, D_n^i\}$ are vertices of $\mathfrak{G}_n(M(G_i))$. The vertices $\{B_1^1 \sqcup B_1^2, \dots, B_n^1 \sqcup B_n^2\}$ and $\{D_1^1 \sqcup B_1^2, \dots, D_n^1 \sqcup B_n^2\}$ are connected because $\mathfrak{G}_n(M(G_1))$ is connected, and $\{D_1^1 \sqcup B_1^2, \dots, D_n^1 \sqcup B_n^2\}$ and $\{D_1^1 \sqcup D_1^2, \dots, D_n^1 \sqcup D_n^2\}$ are connected because $\mathfrak{G}_n(M(G_2))$ is connected. We conclude that any two vertices of $\mathfrak{G}_n(M)$ are connected, so $\mathfrak{G}_n(M)$ is connected.

A key observation is that $M$ has a cocircuit of size $\leq 2n-1$. This is not true in general matroids and this is the most essential way the graphic hypothesis is used. The graph $G$ has a vertex $v$ of degree $\leq 2n-1$ because $G$ has $r+1$ vertices and $nr$ edges, making the average vertex degree $2n\frac{r}{r+1}$. The vertex $v$ is fixed throughout the proof. Let $C$ be the set of edges leaving $v$ and let $N(v)$ be the neighbors of $v$.

Given a vertex $\{B_1, \dots, B_n\}$ of $\mathfrak{G}_n(M)$, define $S_i := B_i \cap C$. We will use this notation for any vertex of $\mathfrak{G}_n(M)$ when there is no ambiguity. We say that $\{B_1, \dots, B_n\}$ is *balanced* if $|S_i| \leq 2$ for each $i$. We first show that each vertex of $\mathfrak{G}_n(M)$ is connected to a balanced vertex. We then show that any two balanced vertices that have the same intersections with $C$ are connected. This is the heart of the proof and where the inductive step is used. Finally, we show that any two balanced vertices are connected. These facts are proved in this order as statements (1), (2), and (3), and these are enough to show that $\mathfrak{G}_n(M)$ is connected.

**(1)** *Every vertex of $\mathfrak{G}_n(M)$ is connected to a balanced vertex.*

Let $\{B_1, \dots, B_n\}$ be a vertex of $\mathfrak{G}_n(M)$. Suppose that $\{B_1, \dots, B_n\}$ is not balanced and (without loss of generality) $|S_1| > 2$. Consider the subgraph $H$ of $G$ with edge set $B_1 - C$ and vertex set $V(G) - v$. It has $|S_1|$ components;

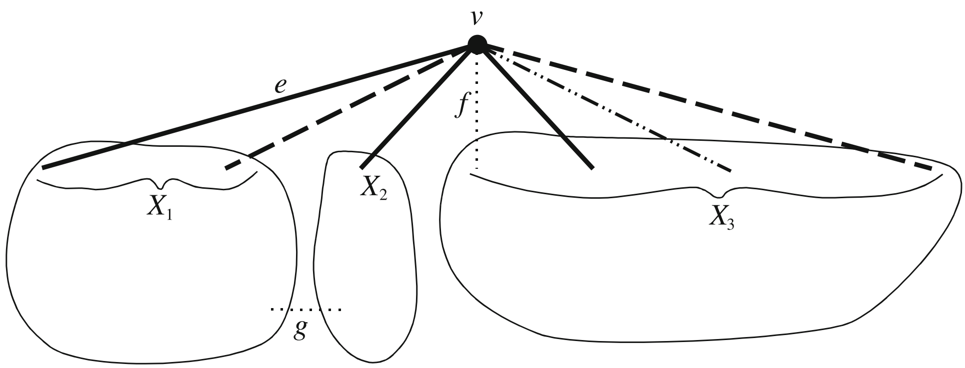


**Figure 1.** Each edge type corresponds to one of the bases $B_i$. The normal edges correspond to $B_1$ and the dotted edges correspond to $B_2$. The blobs represent the components of $H$.

the intersection of these components with $N(v)$ partitions $N(v)$, and therefore $C$, into $|S_1|$ parts. We denote this partition by $X_1 \cup \ldots \cup X_{|S_1|} = C$. See Figure 1. Note that $S_1$ intersects each of the $X_i$ in size 1. As $d(v) \leq 2n-1$, without loss of generality $|S_2|=1$. Say $S_2 = \{f\}$ and $e \in S_1$ is an edge not in the $X_i$ containing $f$ (this is $X_3$ in the figure, and $e \in X_1$; all we need is that $e \notin X_3$). Now double swap $e$ out of $B_1$ and into $B_2$. That is, there exists a $g \in B_2$ such that $B_1 \cup g - e$ and $B_2 \cup e - g$ are bases. The edge $g$ is not in $C$ because $g \in B_2$, $S_2 = \{f\}$, and $f$ and $g$ are distinct: if $f = g$, then $B_1 \cup g - e$ intersects $X_3$ in size 2 contradicting that it's a base. Therefore $|(B_1 \cup g - e) \cap C| = |S_1| - 1$ and $|(B_2 \cup e - g) \cap C| = 2$. By repeating such swaps we eventually obtain a balanced vertex. This proves (1).

Given a balanced vertex $\{B_1, \ldots, B_n\}$, its *matching graph* is the graph with vertex set $C$ and edge set $\{S_i : |S_i| = 2, i \in [n]\}$ (an edge being identified with its two element set of ends). Note that the matching graph has vertices of degree at most one and at least one isolated vertex.

**(2)** *If two balanced vertices of $\mathfrak{G}_n(M)$ have identical matching graphs, then they are connected.*

Let $\{B_1, \ldots, B_n\}$ and $\{D_1, \ldots, D_n\}$ be balanced vertices of $\mathfrak{G}_n(M)$ with the same matching graph, labeled so that $B_i \cap C = D_i \cap C =: S_i$. We obtain a new graph $G'$ from $G$ as follows (see Figure 2): delete $v$ and for each $i$ such that $|S_i| = 2$ add an edge between the vertices of $N(v)$ that are ends of the two edges in $S_i$. The subgraph of $G'$ induced by $N(v)$ is the matching graph of $\{B_1, \ldots, B_n\}$ and we will denote its edge set by $Z$, thinking of this as a subset of $E(G')$. For $i$ such that $|S_i| = 2$, let $e'_i \in Z$ be the edge corresponding to $S_i$. We will refer to the elements of $S_i$ as the *pre-edges* of $e'_i$. Note that

$|Z| \leq n-1$. Bases of $M$ give rise to bases of $M(G')$ as follows:

$$B'_i = \begin{cases} (B_i - C) \cup e'_i, & \text{if } |S_i| = 2 \\ B_i - C, & \text{if } |S_i| = 1. \end{cases}$$

Define $D'_i$ similarly. Here we are using that $\{B_1, \dots, B_n\}$ and $\{D_1, \dots, D_n\}$ have the same matching graph so that the same $G'$ works for the construction of $\{B'_1, \dots, B'_n\}$ and $\{D'_1, \dots, D'_n\}$. For $i$ such that $|S_i| = 2$, the subgraph of $G$ with edge set $B_i$ is obtained from the subgraph of $G'$ with edge set $B'_i$ by subdividing $e'_i$. The $B'_i$ are edge-disjoint spanning trees of $G'$ with union $E(G')$. Moreover, $|B'_i| = |B_i| - 1$ is the rank of $M(G')$ so we can apply the induction hypothesis to conclude that there is a path $P$ from $\{B'_1, \dots, B'_n\}$ to $\{D'_1, \dots, D'_n\}$ in $\mathfrak{G}_n(M(G'))$. We will convert this to a path in $\mathfrak{G}_n(M)$.

Given any set of $n$ disjoint bases of $M(G')$, we can reverse the above process to produce $n$ disjoint bases of $M$: if some base $B'$ of $M(G')$ intersects $Z$ in size $t > 0$, choose $t+1$ of the pre-edges of $B' \cap Z$ so that the resulting union with $B' - Z$ is a base of $M$ (not all choices of $t+1$ edges will work, but at least one will since $B' - Z$ is a forest and $(B' - Z) \cup \text{pre-edges}(B' \cap Z)$ spans $V(G)$). Of the $2t$ elements in pre-edges$(B' \cap Z)$, $t-1$ remain to be assigned to bases of $M$ and these will be assigned to bases not intersecting $Z$. Also, there exists $e^* \in C$ that is not the pre-edge of any $e'_i$. This will always be added to some base not intersecting $Z$. We call this process of taking a base of $M(G')$ and producing a base of $M$ *pulling back*, and we call the base of $M$ the *pull-back* of the base of $M(G')$; we also use this terminology for sets of bases as follows. To pull back a vertex $\{M'_1, \dots, M'_n\}$ of $\mathfrak{G}_n(M(G'))$, pull back the bases intersecting $Z$ first. There are typically many choices for each of these pull-backs, and these choices can be made independently since the sets pre-edges$(M'_i \cap Z)$ are disjoint; let $Y \subseteq C$ be the union of the edges used by each of these pull-backs. Next, pull back the bases not intersecting $Z$ by adding to each a single edge of $C - Y$.

Pull each vertex in the path $P$ back to a vertex of $\mathfrak{G}_n(M)$. It is necessary in what follows to require that these pull-backs are made consistently in the following sense: if a base $B'$ of $M(G')$ appears as an element of any vertex of $P$, and $B'$ intersects $Z$, then require that $B'$ is pulled back to the same base of $M$ each time it is pulled back as an element of a vertex of $P$. This is possible because the choices for pull-backs of bases intersecting $Z$ are made independently, as mentioned above. (We would like to do the same for bases not intersecting $Z$, but this isn't possible. The next paragraph gives a way to get around this.) Now suppose that $\{M'_1, \dots, M'_n\}$ and $\{N'_1, \dots, N'_n\}$ are consecutive vertices in $P$. Without loss of generality, $M'_1 = N'_1$. Let $M_1$ and

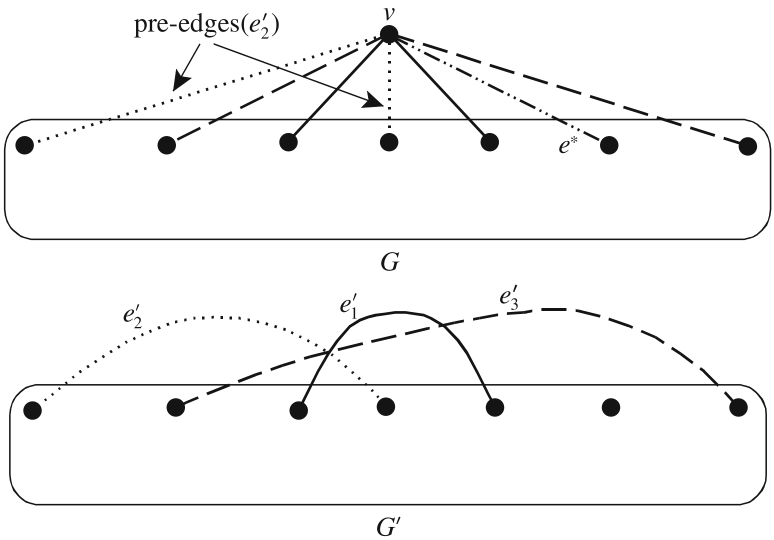


**Figure 2.** Each edge type corresponds to a base of $G$ and $G'$.

$N_1$ be the corresponding pulled back bases of $M$. We want $M_1 = N_1$. If $M'_1$ intersects $Z$, this is true by the consistency condition just mentioned.

If $M'_1$ does not intersect $Z$, $M_1$ may differ from $N_1$ by one element. Suppose $e^* \in M_j$. If $j \neq 1$, double swap $e^*$ of $M_j$ with $M_1 \cap C$ of $M_1$ (this is possible because $|M_j \cap C| = |M_1 \cap C| = 1$). Denote the resulting set of bases by $\{X_1, \ldots, X_n\}$, and put $\{X_1, \ldots, X_n\} = \{M_1, \ldots, M_n\}$ in the case $j = 1$. Do the same thing with $\{N_1, \ldots, N_n\}$ and $N_1$ to obtain $\{Y_1, \ldots, Y_n\}$. The vertices $\{X_1, \ldots, X_n\}$ and $\{Y_1, \ldots, Y_n\}$ are adjacent in $\mathfrak{G}_n(M)$ because $X_1 = M'_1 \cup e^* = N'_1 \cup e^* = Y_1$. Therefore there is a path between $\{M_1, \ldots, M_n\}$ and $\{N_1, \ldots, N_n\}$ in $\mathfrak{G}_n(M)$. This proves that the pulled back path can be patched up to make a walk from $\{B_1, \ldots, B_n\}$ to $\{D_1, \ldots, D_n\}$ in $\mathfrak{G}_n(M)$. This proves (2).

**(3)** *If $\{B_1, \ldots, B_n\}$ and $\{D_1, \ldots, D_n\}$ are balanced vertices of $\mathfrak{G}_n(M)$, then there is a balanced vertex $\{M_1, \ldots, M_n\}$ connected to $\{B_1, \ldots, B_n\}$ and a balanced vertex $\{N_1, \ldots, N_n\}$ connected to $\{D_1, \ldots, D_n\}$ such that $\{M_1, \ldots, M_n\}$ and $\{N_1, \ldots, N_n\}$ have the same matching graph.*

First note that (2) and (3) together show that any two balanced vertices are connected, and therefore proving (3) will complete the proof of Theorem 3.1. We prove (3) by rearranging the parts of the bases that intersect $C$ without changing the other parts. Although the proof is rather involved, it is not hard to convince oneself that the result is true by trying small values

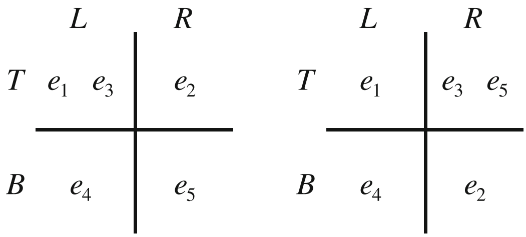


**Figure 3.** Two possibilities for positions of $e_1,\dots,e_5$ in the partitions $L\cup R, T\cup B = V(H)$. In the example on the left, moves (i), (iii), and (vi) are valid. In the example on the right, moves (i), (ii), and (iv) are valid.

of $n$. Proving the result for $n=3$ and $d(v)=4$ only requires checking a few cases.

A *valid move* on a matching graph $H$ of a balanced vertex $\{B_1,\dots,B_n\}$ is a change in the matching graph from $H$ to $H'$ such that there is a balanced vertex connected to $\{B_1,\dots,B_n\}$ with matching graph $H'$. First we show the existence of certain valid moves and then we show that these are enough to prove (3).

**(A)** *Suppose $\{B_1,\dots,B_n\}$ is a balanced vertex with matching graph $H$, $(e_1,e_2)$ and $(e_3,e_4)$ are edges of $H$, and $e_5$ is an isolated vertex. Then at least one of (i) and (ii) and (isomorphically) at least one of (iii) and (iv) are valid moves on $H$. Furthermore, if (v) and (vi) are not valid moves, then either (i) and (ii) are both valid or (iii) and (iv) are both valid.*

*(i) Deleting $(e_1,e_2)$ and adding $(e_1,e_5)$.*
*(ii) Deleting $(e_1,e_2)$ and adding $(e_2,e_5)$.*
*(iii) Deleting $(e_3,e_4)$ and adding $(e_3,e_5)$.*
*(iv) Deleting $(e_3,e_4)$ and adding $(e_4,e_5)$.*
*(v) Deleting $\{(e_1,e_2),(e_3,e_4)\}$ and adding $\{(e_1,e_3),(e_2,e_4)\}$.*
*(vi) Deleting $\{(e_1,e_2),(e_3,e_4)\}$ and adding $\{(e_1,e_4),(e_2,e_3)\}$.*

We work again with double swaps. Suppose $S_1=e_1\cup e_2$, $S_2=e_3\cup e_4$, and $S_3=e_5$. Recall from the proof of (1) that $B_1$ and $B_2$ determine partitions of $C$ into two parts. Suppose that $B_1$ determines the partition $L\cup R=C=V(H)$ and $B_2$ determines $T\cup B=C=V(H)$. For $X\subset C$, $X\cup(B_1-C)$ (resp. $X\cup(B_2-C)$) is a base if and only if $X$ intersects $L$ and $R$ (resp. $T$ and $B$) in size 1. Now to show the first part of (A), double swap $e_5\in B_3$ into $B_1$. The edge $e_5$ must be swapped with something in $C$, so either (i) or (ii) holds. The same argument shows that (iii) or (iv) holds.

Consider the representation of the partitions $L\cup R=T\cup B=V(H)$ as shown in Figure 3. The four regions correspond to the sets $L\cap T$, $L\cap B$,

$R \cap T$, and $R \cap B$. If one of the regions contains two of $e_1, e_2, e_3, e_4$ (as in the left example), then these elements can be double swapped with each other. This means we can replace $(e_1, e_2), (e_3, e_4)$ by either $(e_1, e_3), (e_2, e_4)$ or $(e_1, e_4), (e_2, e_3)$ in the matching graph and the resulting matching graph is realized by some vertex connected to $\{B_1, \ldots, B_n\}$; either (v) or (vi) is a valid move. If none of the regions contains two of $e_1, e_2, e_3, e_4$, then we are in a situation isomorphic to the right example of Figure 3. In this case (i) holds because $B_1 \cup e_5 - e_2$ and $B_3 \cup e_2 - e_5$ are bases, but in addition (ii) holds. This is because there is a path in $\mathfrak{G}_n(M)$ with consecutive vertices $\{B_1, \ldots, B_n\}$, $\{B_1 \cup \{e_3, e_4\} - \{e_1, e_2\}, B_2 \cup \{e_1, e_2\} - \{e_3, e_4\}, B_3, \ldots, B_n\}$, $\{B_1 \cup \{e_3, e_4\} - \{e_1, e_2\}, B_2 \cup \{e_2, e_5\} - \{e_3, e_4\}, B_3 \cup e_1 - e_5, \ldots, B_n\}$. This last vertex has a matching graph that is the same as $H$ except with $(e_1, e_2)$ replaced by $(e_2, e_5)$. If $e_5$ is in another region, (iii) and (iv) may hold instead of (i) and (ii). This proves (A).

**(B)** *Let $H$ and $H'$ be graphs on the same vertex set both with maximum vertex degree 1 and the same number, $t$, of isolated vertices, where $t > 0$. It is possible to get from $H$ to $H'$ by a sequence of valid moves of the kind described in (A).*

We prove this by induction on $|V(H)|$. The base case is when $H$ and $H'$ are both a single vertex. Let $I_H$ be the set of vertices that can be made isolated in $H$ after at most one valid move (two vertices in this set don't have to be able to be isolated at the same time). Define $I_{H'}$ similarly. Using the moves (i) and (ii), we see that $|I_H|, |I_{H'}| \geq t + |E(H)|$. Since $t + |E(H)| > |V(H)|/2$, there is a vertex $x$ in $I_H \cap I_{H'}$. By possibly redefining $H$ to be a graph one move away from $H$ (and/or by redefining $H'$ to be a graph one move away from $H'$), we may assume that $x$ is isolated in $H$ and $H'$. If $t > 1$, delete $x$ from $H$ and $H'$, and the result follows by induction.

The case $t = 1$ remains. Consider the valid moves that make $x$ the end of an edge: let $N_H$ be the set of vertices that can pair up with $x$ after one move on $H$, and define $N_{H'}$ similarly. We have $|N_H|, |N_{H'}| \geq |E(H)|$. If a move of type (i) and of type (ii) are valid on $H$ (where $x$ takes the role of $e_5$ from (A)) then $|N_H| > |E(H)|$, and therefore there exists $y \in (N_H \cap N_{H'})$. Next, make the moves so that both graphs have the common edge $(x, y)$. Delete this edge from both graphs and the result follows by induction. For the rest of the proof we may assume that $x$ stays isolated and for each edge $(e_1, e_2)$ of $H$, the moves of type (i) and (ii) are not both valid. This implies that for every pair of edges in $H$, either (v) or (vi) is a valid move. By the same argument, we may also assume this for $H'$ and for any graph we reach from $H$ or $H'$ by a sequence of valid moves that keeps $x$ isolated.

For the rest of proof, we modify the statement we are proving by induction: we no longer require the graphs to have an isolated vertex, but for each pair of edges either (v) or (vi) is valid. We will prove this statement for the graphs $H-x$ and $H'-x$. Consider the graph $J$ with vertex set $V(H)-x$ and edge set $E(H)\cup E(H')$. It is 2-regular, and therefore a disjoint union of cycles. If there is more than one component, split up $V(H)-x$ according to the components and win by induction. The remaining case is if $J$ is a cycle. If $J$ is a 2-cycle, $H=H'$. If $|V(J)|\geq 4$, it suffices to consider 5 consecutive vertices $e_1,e_2,e_3,e_4,e_5$ as in Figure 4 ($e_1=e_5$ if $J$ is a 4-cycle, but the proof still works). If replacing $(e_1,e_2)$ and $(e_3,e_4)$ by $(e_1,e_4)$ and $(e_2,e_3)$ is a valid move on $H'$, we get a 2-cycle and win by induction (as in the top graph of Figure 4). The same thing happens if replacing $(e_2,e_3)$ and $(e_4,e_5)$ by $(e_2,e_5)$ and $(e_3,e_4)$ is a valid move on $H$ (as in the bottom graph). If neither of these is a valid move, then (replacing $(e_1,e_2)$ and $(e_3,e_4)$ by $(e_1,e_3)$ and $(e_2,e_4)$) and (replacing $(e_2,e_3)$ and $(e_4,e_5)$ by $(e_2,e_4)$ and $(e_3,e_5)$) are valid moves (as in the right graph). Delete the ends of the common edge $(e_2,e_4)$ and win by induction. This proves (B).

(B) implies (3) by letting $H$ be the matching graph of $\{B_1,\dots,B_n\}$ and $H'$ be the matching graph of $\{D_1,\dots,D_n\}$. A sequence of valid moves beginning with $H$ yields a path from $\{B_1,\dots,B_n\}$ to $\{M_1,\dots,M_n\}$ and a sequence of valid moves beginning with $H'$ yields a path from $\{D_1,\dots,D_n\}$ to $\{N_1,\dots,N_n\}$. (B) says that we can find moves so that $\{M_1,\dots,M_n\}$ and $\{N_1,\dots,N_n\}$ have the same matching graph. ∎

## 4. Quadrics are generated by one element exchanges

The following theorem together with the modified version of Proposition 2.1 shows that Conjecture 1.8 holds for graphic matroids. This will complete the proof of Theorem 1.2.

**Theorem 4.1.** *If $M$ is a graphic matroid of rank $r$ on a ground set of size $2r$, then the single exchange graph $\mathfrak{G}(M)$ is connected.*

**Proof.** The proof is very similar to the proof of Theorem 3.1. We do induction on $r$. We can assume $G$ is connected using the same argument as before. And again, we have that there is a vertex $v$ of degree at most 3, which we fix throughout the proof. Let $C$ be the set of edges leaving $v$. There is no need to balance the vertices because they already are balanced (if $d(v)=3$, one base intersects $C$ in size 2 and the other in size 1; if $d(v)=2$, both

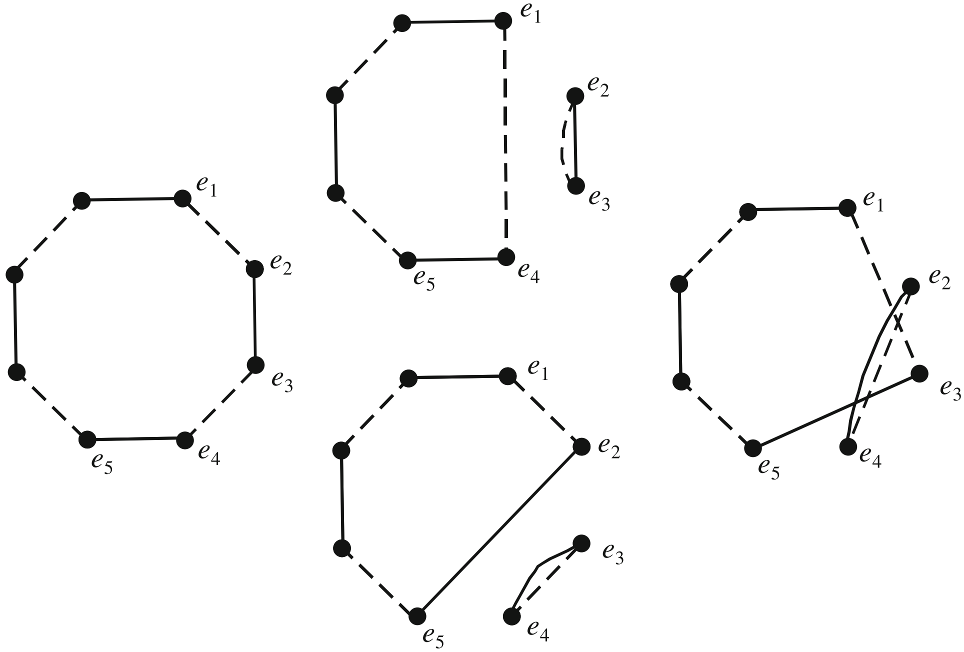


**Figure 4.** On the left is $J$, where dashed edges are edges of $H'$ and normal edges are edges of $H$. The other graphs show $J$ after possible moves on $H$ and $H'$ as described in the proof of (B).

bases intersect $C$ in size 1; if $d(v)=1$, $\mathfrak{G}(M)$ is empty). As before, define the matching graph of a vertex $(B_1,B_2)$ to be the graph with vertex set $C$ and edge set $\{S_i : |S_i|=2, i\in[n]\}$. Note that the matching graph ignores the order of $B_1$ and $B_2$; we are careful to remember this when proving (1) below.

We need to show that any two vertices of $\mathfrak{G}(M)$ that have the same matching graph are connected. This is enough to prove the theorem since statement (3) of the proof of Theorem 3.1 holds with $\mathfrak{G}_n(M)$ replaced by $\mathfrak{G}(M)$. All that is needed to prove this are the moves of type (i) and (ii) of (3)(A), which only use one double swap, and the argument from the first paragraph of the proof of (3)(B).

**(1)** *If two vertices $(B_1,B_2),(D_1,D_2)$ of $\mathfrak{G}(M)$ have the same matching graph, then they are connected.*

We obtain a new graph $G'$ from $G$ using the same construction as in the proof of Theorem 3.1. Let $B'_1,B'_2,D'_1,D'_2$ be the bases of $M(G')$ defined in the proof of Theorem 3.1. By induction on $r$, there is a path from $(B'_1,B'_2)$ to $(D'_1,D'_2)$ in $\mathfrak{G}(M(G'))$. We will convert this to a path in $\mathfrak{G}(M)$. Note that in this case the pull-backs are unique.

Pull each vertex in the path from $(B_1', B_2')$ to $(D_1', D_2')$ back to a vertex of $\mathfrak{G}(M)$. Now suppose that $(M_1', M_2')$ and $(N_1', N_2')$ are consecutive vertices in the path in $\mathfrak{G}(M(G'))$. Let $M_1, M_2, N_1, N_2$ be the corresponding pulled back bases of $M$. If $d(v) = 2$, $(M_1, M_2)$ is adjacent to $(N_1, N_2)$ and we are done. If $d(v) = 3$, observe that $Z$ consists of a single edge $e_1'$ and $C =$ pre-edges$(e_1') \cup e^*$. Without loss of generality, $e_1' \in M_1'$. If $e_1' \in N_1'$, then $M_1$ and $N_1$ differ by only one element and are therefore adjacent in $\mathfrak{G}(M)$ (note that $|M_1 \cap N_1| = r-1$ implies $|M_2 \cap N_2| = r-1$ since $M_1 \sqcup M_2 = N_1 \sqcup N_2$).

If $e_1' \in N_2'$, then let $\{a, b\} =$ pre-edges$(e_1')$. Double swap $e^* \in M_2$ with $M_1$ to obtain a vertex $(P_1, P_2)$ adjacent to $(M_1, M_2)$. The edge $e^*$ must double swap with something in $C$ (say, $a$), because otherwise $P_2$ would not intersect $C$ contradicting that it's a base. We know that $(N_2' - e_1') \subset M_2'$ and we can rewrite this as $(N_2 - \{a, b\}) \subset (M_2 - e^*)$. Add $\{a, b\}$ to both these sets to obtain $N_2 \subset (P_2 \cup b)$ and therefore $|P_2 \cap N_2| = r - 1$. This shows that $(P_1, P_2)$ and $(N_1, N_2)$ are adjacent and thus the pulled back path can be patched up to make a walk from $(B_1, B_2)$ to $(D_1, D_2)$ in $\mathfrak{G}(M)$. ∎

## 5. Future Work

The proofs of Theorems 3.1 and 4.1 depend heavily on the graphic assumption. However, it seems possible to convert many of the techniques to the general case. For instance, instead of choosing $C$ to be the edges leaving a vertex, we could take $C$ to be a cocircuit. There is an analog of the construction of $G'$ for any cocircuit of a matroid. One thing that can definitely not be generalized is the existence of a small cocircuit and this is crucial to the proofs. For instance, there are uniform matroids with arbitrarily large minimum cocircuit size for fixed $n$.

Part (3) of Theorem 3.1 at first seemed like a digression from the main content of the proof and theorem, and a fun, but not very significant, result on its own. However, the analogous statement of (3) for general cocircuits may actually be rather deep. We will not state the exact generalization of (3), but it suggests the following question. Given matroids $M$ and $N$ on the ground set $E$ and $X \subset E$, define $r_{M\cap N}(X)$ to be the maximum size of an independent set in $X$ common to $M$ and $N$. Given matroids $M_1, \ldots, M_n$ and $N_1, \ldots, N_n$ all on the ground set $E$, we define their *matching intersection rank* to be

$$\max_{\pi \in S_n} \Big( \max_{X_1 \sqcup \ldots \sqcup X_n = E} \sum_{i=1}^{n} r_{M_i \cap N_{\pi(i)}}(X_i) \Big).$$

**Problem.** Suppose $\{B_1,\dots,B_n\}$ and $\{D_1,\dots,D_n\}$ are balanced vertices of $\mathfrak{G}_n(M)$ with respect to some cocircuit $C$. Here we take balanced to mean that the intersection sizes of the bases with $C$ are less than one away from average intersection size. Let $M_i = M.(B_i - C)|C$ and $N_i = M.(D_i - C)|C$, where . denotes contraction and $|C$ means deleting everything not in $C$. Determine conditions on $C$ under which the matching intersection rank of $M_1,\dots,M_n$ and $N_1,\dots,N_n$ is $|C|$.

This problem and the notion of matching intersection rank lead to two general questions, but we have not been able to formulate specific conjectures along these lines. Is there a generalization of the matroid union and intersection theorems that says something about matching intersection rank? Does White's conjecture generalize to a statement that involves bases of more than one matroid?

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Jonah Blasiak

*Department of Mathematics*
*University of California, Berkeley*
*970 Evans Hall*
*Berkeley, CA 94720-3840*
*USA*
jblasiak@berkeley.edu